A Unified and Complete Construction of All Finite Dimensional Irreducible Representations of $gl(2|2)$

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Abstract

Representations of the non-semisimple superalgebra $gl(2|2)$ in the standard basis are investigated by means of the vector coherent state method and boson-fermion realization. All finite-dimensional irreducible typical and atypical representations and lowest weight (indecomposable) Kac modules of $gl(2|2)$ are constructed explicitly through the explicit construction of all $gl(2) \oplus gl(2)$ particle states (multiplets) in terms of boson and fermion creation operators in the super-Fock space. This gives a unified and complete treatment of finite-dimensional representations of $gl(2|2)$ in explicit form, essential for the construction of primary fields of the corresponding current superalgebra at arbitrary level.

I Introduction

Recently there is much research interest in superalgebras and their corresponding non-unitary conformal field theories (CFTs), because of their applications in high energy and condensed matter physics including topological field theory [1, 2], logarithmic CFTs (see e.g. [3] and references therein), disordered systems and the integer quantum Hall effects [4, 5, 6, 7, 8, 9, 10, 11]. In such contexts, the vanishing of superdimensions and Virasoro central charges and the existence of primary fields with negative dimensions are crucial [5, 6]. The most interesting algebras with such properties are $osp(n|n)$ and $gl(n|n)$.

In most physical applications, one needs the explicit construction of finite-dimensional representations of a superalgebra. This is particularly the case in superalgebra CFTs. To construct primary fields of such CFTs in terms of free fields, one has to construct the finite-dimensional representations of the superalgebras explicitly. The explicit construction of the primary fields is essential in the investigation of disordered systems by the supersymmetric method.

Unlike ordinary bosonic algebras, there are two types of representations for most superalgebras. They are the so-called typical and atypical representations. The typical representations are irreducible and are similar to the usual representations appeared in ordinary bosonic algebras. The atypical representations have no counterpart in the bosonic algebra setting. They can be irreducible or not fully reducible (i.e. reducible or indecomposable). This makes the study of representations of superalgebras very difficult.

Representations of $osp(2|2)$ were investigated in [12, 13]. A unified construction of finite-dimensional typical and atypical representations of $osp(2|2)$ were given in [14, 15] by means of the vector coherent state method. This enabled the explicit construction of all primary fields of the $osp(2|2)$ CFT [16, 14] in terms of free fields [17, 18].

In this paper we investigate finite-dimensional representations of the non-semisimple superalgebra $gl(2|2)$. All finite-dimensional irreducible typical and atypical representations and lowest weight (indecomposable) Kac modules of $gl(2|2)$ are constructed explicitly through the explicit construction of all $gl(2)\oplus gl(2)$ particle states (multiplets) in terms of the boson and fermion creation operators in the super-Fock space. This we believe gives a unified and complete treatment of all finite-dimensional irreducible representations of $gl(2|2)$ in explicit form.

Let us point out that the finite-dimensional representations of $gl(2|2)$ have also been investigated in [19, 20] using the GT basis. Our method is completely different from and in our opinion is simpler than the method used in these two references. Moreover our results can be used to construct primary fields of the corresponding $gl(2|2)$ CFTs at arbitrary level, which is the subject of a separate work.

This paper is organized as follows. In section 2, we introduce our notations and derive a free boson-fermion realization of $gl(2|2)$ by means of the vector coherent state method. In section 3, we describe the explict construction of independent $gl(2) \oplus gl(2)$ particle states in the super-Fock space. We derive the actions of odd simple generators of $gl(2|2)$ on these multiplets. The 16 independent multiplets constructed span all finite-dimensional irreducible typical representations of $gl(2|2)$. In section 4, we deduce and construct all four types of finite-dimensional irreducible atypical representations and lowest weight (indecomposable) Kac modules of $gl(2|2)$.

II Boson-fermion Realization of $gl(2|2)$

In this section, we obtain a boson-fermion realization of the superalgebra $gl(2|2)$ in the standard basis.

This superalgebra is non-semisimple and can be written as $gl(2|2) = gl(2|2)^{\rm even} \oplus gl(2|2)^{\rm odd}$, where

$$gl(2|2)^{\rm even} = gl(2) \oplus gl(2)$$

$$\begin{aligned} &= \{I\} \oplus \{\{E_{12}, E_{21}, H_1\} \oplus \{E_{34}, E_{43}, H_2\}, N\}, \\ gl(2|2)^{\rm odd} &= \{E_{13}, E_{31}, E_{23}, E_{32}, E_{24}, E_{42}, E_{14}, E_{41}\}. \end{aligned} \qquad \text{(II.1)}$$

In the standard basis, E_{12}, E_{34}, E_{23} (E_{21}, E_{43}, E_{32}) are simple raising (lowering) generators, E_{13}, E_{14}, E_{24} (E_{31}, E_{41}, E_{42}) are non-simple raising (lowering) generators and H_1, H_2, I, N are elements of the Cartan subalgebra. We have

$$\begin{aligned} H_1 &= E_{11} - E_{22}, \qquad H_2 = E_{33} - E_{44}, \\ I &= E_{11} + E_{22} + E_{33} + E_{44}, \\ N &= E_{11} + E_{22} - E_{33} - E_{44} + \beta I \end{aligned} \qquad \text{(II.2)}$$

with β being an arbitrary parameter. That N is not uniquely determined is a consequence of the fact that $gl(2|2)$ is non-semisimple. The generators obey the following (anti-)commutation relations:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-1)^{([i]+[j])([k]+[l])} \delta_{il} E_{kj}, \qquad \text{(II.3)}$$

where $[E_{ij}, E_{kl}] \equiv E_{ij}E_{kl} - (-1)^{([i]+[j])([k]+[l])} E_{kl}E_{ij}$ is a commutator or an anticommutator, $[1] = [2] = 0,\ [3] = [4] = 1$, and $E_{ii},\ i = 1,2,3,4$ are related to H_1, H_2, I, N via (II.2). The quadratic Casimir of the algebra is given by $C_2 = \sum_{AB} (-1)^{[B]} E_{AB} E_{BA}$.

Let $|hw>$ be the highest weight state of highest weight (J_1, J_2, q, p) of $gl(2|2)$ defined by

$$\begin{aligned} &H_1|hw> = 2J_1|hw>, \quad H_2|hw> = 2J_2|hw>, \\ &I|hw> = 2q|hw>, \quad N|hw> = 2p|hw>, \\ &E_{12}|hw> = E_{34}|hw> = E_{23}|hw> = E_{13}|hw> = E_{14}|hw> = E_{24}|hw> = 0. \end{aligned} \qquad \text{(II.4)}$$

Here J_1, J_2 are positive integers and half-integers and q, p are arbitrary complex numbers. Define the coherent state [21, 22]

$$e^{E_{21}a_{12} + E_{43}a_{34} + E_{31}\alpha_{13} + E_{32}\alpha_{23} + E_{42}\alpha_{24} + E_{41}\alpha_{14}}|hw> .$$

Then state vectors are mapped into functions

$$\psi_{J_1,J_2,q,p} = < hw| e^{\alpha_{13}^\dagger E_{13} + \alpha_{23}^\dagger E_{23} + \alpha_{24}^\dagger E_{24} + \alpha_{14}^\dagger E_{14} + a_{12}^\dagger E_{12} + a_{34}^\dagger E_{34}} |\psi> |0>, \qquad \text{(II.5)}$$

and operators A are mapped as follows

$$A|\psi> \longrightarrow \Gamma(A)\psi_{J_1,J_2,q,p} = < hw| e^{\alpha_{13}^\dagger E_{13} + \alpha_{23}^\dagger E_{23} + \alpha_{24}^\dagger E_{24} + \alpha_{14}^\dagger E_{14} + a_{12}^\dagger E_{12} + a_{34}^\dagger E_{34}} A|\psi> |0> . \qquad \text{(II.6)}$$

Here $\alpha_{ij}^\dagger$ (α_{ij}) are fermion operators with number operators $N_{\alpha_{ij}}$ and $a_{ij}^\dagger$ (a_{ij}) are boson operators with number operators $N_{a_{ij}}$. They obey relations

$$\begin{aligned}
&\{\alpha_{ij}, \alpha_{kl}^\dagger\} = \delta_{ik}\delta_{jl}, \qquad (\alpha_{ij})^2 = \left(\alpha_{ij}^\dagger\right)^2 = 0,\\
&[N_{\alpha_{ij}}, \alpha_{kl}] = -\delta_{ik}\delta_{jl}\alpha_{kl}, \qquad [N_{\alpha_{ij}}, \alpha_{kl}^\dagger] = \delta_{ik}\delta_{jl}\alpha_{kl}^\dagger,\\
&[a_{ij}, a_{kl}^\dagger] = \delta_{ik}\delta_{jl},\\
&[N_{a_{ij}}, a_{kl}] = -\delta_{ik}\delta_{jl}a_{kl}, \qquad [N_{a_{ij}}, a_{kl}^\dagger] = \delta_{ik}\delta_{jl}a_{kl}^\dagger,
\end{aligned} \tag{II.7}$$

and all other (anti-)commutators vanish. Moreover, $a_{12}|0>= a_{34}|0>= \alpha_{23}|0>= \alpha_{13}|0>= \alpha_{14}|0>= \alpha_{24}|0>= 0$.

Taking E_{12}, E_{34} etc in turn and after long algebraic computations, we find the following representation of simple generators in terms of the boson and fermion operators:

$$\begin{aligned}
\Gamma(E_{12}) &= a_{12} - \frac{1}{2}\alpha_{23}^\dagger\alpha_{13} + \frac{1}{2}\left(\frac{1}{6}a_{34}^\dagger\alpha_{23}^\dagger - \alpha_{24}^\dagger\right)\alpha_{14},\\
\Gamma(E_{34}) &= a_{34} + \frac{1}{2}\alpha_{23}^\dagger\alpha_{24} + \frac{1}{2}\left(\frac{1}{6}a_{12}^\dagger\alpha_{23}^\dagger + \alpha_{13}^\dagger\right)\alpha_{14},\\
\Gamma(E_{23}) &= \alpha_{23} + \frac{1}{2}a_{12}^\dagger\alpha_{13} - \frac{1}{2}a_{34}^\dagger\left(\alpha_{24} + \frac{1}{3}a_{12}^\dagger\alpha_{14}\right),\\
\Gamma(H_1) &= 2J_1 - 2N_{a_{12}} + N_{\alpha_{23}} - N_{\alpha_{13}} + N_{\alpha_{24}} - N_{\alpha_{14}},\\
\Gamma(H_2) &= 2J_2 - 2N_{a_{34}} + N_{\alpha_{23}} + N_{\alpha_{13}} - N_{\alpha_{24}} - N_{\alpha_{14}},\\
\Gamma(I) &= 2q,\\
\Gamma(N) &= 2p - 2\left(N_{\alpha_{23}} + N_{\alpha_{13}} + N_{\alpha_{24}} + N_{\alpha_{14}}\right),\\
\Gamma(E_{21}) &= a_{12}^\dagger\left[2J_1 - N_{a_{12}} + \frac{1}{2}\left(N_{\alpha_{23}} - N_{\alpha_{13}} + N_{\alpha_{24}} - N_{\alpha_{14}}\right)\right]\\
&\quad -\alpha_{13}^\dagger\alpha_{23} - \alpha_{14}^\dagger\alpha_{24} - \frac{1}{4}\left(a_{12}^\dagger\right)^2\alpha_{23}^\dagger\alpha_{13}\\
&\quad +\frac{1}{12}a_{12}^\dagger a_{34}^\dagger\alpha_{23}^\dagger\alpha_{24} - \frac{1}{4}a_{12}^\dagger\left(a_{12}^\dagger\alpha_{24}^\dagger + \frac{1}{3}a_{34}^\dagger\alpha_{13}^\dagger\right)\alpha_{14},\\
\Gamma(E_{43}) &= a_{34}^\dagger\left[2J_2 - N_{a_{34}} + \frac{1}{2}\left(N_{\alpha_{23}} + N_{\alpha_{13}} - N_{\alpha_{24}} - N_{\alpha_{14}}\right)\right]\\
&\quad +\alpha_{24}^\dagger\alpha_{23} + \alpha_{14}^\dagger\alpha_{13} + \frac{1}{4}\left(a_{34}^\dagger\right)^2\alpha_{23}^\dagger\alpha_{24}\\
&\quad -\frac{1}{12}a_{12}^\dagger a_{34}^\dagger\alpha_{23}^\dagger\alpha_{13} + \frac{1}{4}\left(a_{34}^\dagger\alpha_{12}^\dagger + \frac{1}{3}a_{12}^\dagger\alpha_{24}^\dagger\right)a_{34}^\dagger\alpha_{14},\\
\Gamma(E_{32}) &= \alpha_{23}^\dagger\left[q - J_1 + J_2 + \frac{1}{2}(N_{a_{12}} - N_{a_{34}} + N_{\alpha_{13}} - N_{\alpha_{24}})\right]\\
&\quad +\alpha_{13}^\dagger a_{12} + \alpha_{24}^\dagger a_{34} + \frac{1}{6}\alpha_{23}^\dagger(a_{12}^\dagger\alpha_{24}^\dagger + a_{34}^\dagger\alpha_{13}^\dagger)\alpha_{14},
\end{aligned} \tag{II.8}$$

and the representation for non-simple generators is easily obtained from that of simple generators above by means of the commutation relations. (II.8) gives a boson-fermion realization of the non-semisimple superalgebra $gl(2|2)$ in the standard basis. In this realization, the Casimir takes a constant value: $C_2 = 2[(J_1 - J_2)(J_1 + J_2 + 1) + q(p - 2)]$.

III Typical Representations of $gl(2|2)$

Representations of $gl(2|2)$ are labelled by (J_1, J_2, q, p) with J_1, J_2 being positive integers or half-integers and q, p being arbitrary complex numbers. Consider a particle state in the super-Fock space, obtained by acting the creation operators on the vacuum vector $|0>$. We call such a state a level-x state if $\Gamma(H_1), \Gamma(H_2), \Gamma(I), \Gamma(N)$ have eigenvalues $2(m_1+x), 2(m_2+x), 2q, 2(p-x)$, respectively. Obviously, a level-x state is a product of x number of fermion creation operators and boson creation operators of the form $\left(a_{12}^\dagger\right)^{J_1-m_1-y}\left(a_{34}^\dagger\right)^{J_2-m_2-\bar{y}}$ acting on $|0>$, where $y, \bar{y}$ are certain integers or half-integers, depending on the values of x. It is easy to see that there are 16 independent such states obtained from 16 independent combinations of the creation operators. This includes one level-0 state, four level-1 states, six level-2 states, four level-3 states and one level-4 state. Thus each $gl(2|2)$ representation decomposes into at most 16 representations of the even subalgebra $gl(2)\oplus gl(2)$. Let us construct representations for $gl(2)\oplus gl(2)$ out of the above states. First the level-0 and level-4 states are already representations of $gl(2)\oplus gl(2)$ with highest weights (J_1, J_2, q, p) and $(J_1, J_2, q, p-4)$ respectively. We denote these two multiplets by $|J_1, m_1, J_2, m_2, q; p>$ and $|J_1, m_1, J_2, m_2, q; p-4>$, respectively. So

$$\begin{aligned}
&|J_1, m_1, J_2, m_2, q; p> = \left(a_{12}^\dagger\right)^{J_1-m_1}\left(a_{34}^\dagger\right)^{J_2-m_2}|0>,\\
&\quad m_1 = J_1, J_1-1, \cdots, -J_1, \quad m_2 = J_2, J_2-1, \cdots, -J_2,\\
&|J_1, m_1, J_2, m_2, q; p-4> = \alpha_{23}^\dagger\alpha_{13}^\dagger\alpha_{24}^\dagger\alpha_{14}^\dagger\left(a_{12}^\dagger\right)^{J_1-m_1-4}\left(a_{34}^\dagger\right)^{J_2-m_2-4}|0>,\\
&\quad m_1 = J_1-4, J_1-5, \cdots, -(J_1+4), \quad m_2 = J_2-4, J_2-5, \cdots, -(J_2+4)
\end{aligned}\tag{III.1}$$

Both multiplets have dimension $(2J_1+1)(2J_2+1)$.

It can be shown that other level-x states can be combined into independent level-x multiplets of $gl(2)\oplus gl(2)$ with certain highest weights. The procedure is the following. For a given level x, one considers a combination Ψ_{J_1,m_1,J_2,m_2} of all level-x states. The combination coefficients are in general functions of J_1, m_1, J_2, m_2. We require that Ψ_{J_1,m_1,J_2,m_2} be a representation of $gl(2)\oplus gl(2)$. In order for the representation to be finite-dimensional, the actions of the $gl(2)\oplus gl(2)$ generators on Ψ_{J_1,m_1,J_2,m_2} must have the following form:

$$\begin{aligned}
\Gamma(E_{12})\Psi_{J_1,m_1,J_2,m_2} &= (J_1-m_1-z)\Psi_{J_1,m_1+1,J_2,m_2},\\
\Gamma(E_{21})\Psi_{J_1,m_1,J_2,m_2} &= (J_1+m_1+\bar{z})\Psi_{J_1,m_1-1,J_2,m_2},\\
\Gamma(E_{34})\Psi_{J_1,m_1,J_2,m_2} &= (J_2-m_2-u)\Psi_{J_1,m_1,J_2,m_2+1},\\
\Gamma(E_{43})\Psi_{J_1,m_1,J_2,m_2} &= (J_2+m_2+\bar{u})\Psi_{J_1,m_1,J_2,m_2-1},
\end{aligned}\tag{III.2}$$

where $z, \bar{z}, u, \bar{u}$ are some integers or half integers to be determined together with the combination coefficients. These requirements give rise to difference equations for the combination coefficients. Solving these difference equations simutaneously for each level x, we

determine the combination coefficients and $z, \bar{z}, u, \bar{u}$. The procedure of solving the difference equations for each level x is non-trivial and requires long algebraic manipulations. Here we omit the details and only list the results as follows.

The four level-1 states can be combined into four independent multiplets of $gl(2)\oplus gl(2)$ with highest weights $(J_1-\frac{1}{2}, J_2-\frac{1}{2}, q, p-1)$, $(J_1+\frac{1}{2}, J_2-\frac{1}{2}, q, p-1)$, $(J_1+\frac{1}{2}, J_2+\frac{1}{2}, q, p-1)$ and $(J_1-\frac{1}{2}, J_2+\frac{1}{2}, q, p-1)$, respectively:

$$
\begin{aligned}
&|J_1-\frac{1}{2}, m_1, J_2-\frac{1}{2}, m_2, q; p-1>=(\alpha^\dagger_{14}+\frac{1}{2}a^\dagger_{12}\alpha^\dagger_{24}-\frac{1}{2}\alpha^\dagger_{13}a^\dagger_{34}-\frac{1}{3}a^\dagger_{12}\alpha^\dagger_{23}a^\dagger_{34})\\
&\quad\times\left(a^\dagger_{12}\right)^{J_1-m_1-\frac{3}{2}}\left(a^\dagger_{34}\right)^{J_2-m_2-\frac{3}{2}}|0>,\quad J_1, J_2\geq\frac{1}{2},\\
&\quad m_1=J_1-\frac{3}{2}, J_1-\frac{5}{2},\cdots,-(J_1+\frac{1}{2}),\quad m_2=J_2-\frac{3}{2}, J_2-\frac{5}{2},\cdots,-(J_2+\frac{1}{2}),\\
&|J_1+\frac{1}{2}, m_1, J_2-\frac{1}{2}, m_2, q; p-1>\\
&\;=\Big[\frac{1}{2}(3J_1+m_1+\frac{5}{2})a^\dagger_{12}\alpha^\dagger_{24}-\frac{1}{3}(2J_1+m_1+2)a^\dagger_{12}\alpha^\dagger_{23}a^\dagger_{34}\\
&\quad-(J_1-m_1-\frac{1}{2})(\alpha^\dagger_{14}-\frac{1}{2}\alpha^\dagger_{13}a^\dagger_{34})\Big]\left(a^\dagger_{12}\right)^{J_1-m_1-\frac{3}{2}}\left(a^\dagger_{34}\right)^{J_2-m_2-\frac{3}{2}}|0>,\quad J_2\geq\frac{1}{2},\\
&\quad m_1=J_1-\frac{1}{2}, J_1-\frac{3}{2},\cdots,-(J_1+\frac{3}{2}),\quad m_2=J_2-\frac{3}{2}, J_2-\frac{5}{2},\cdots,-(J_2+\frac{1}{2}),\\
&|J_1+\frac{1}{2}, m_1, J_2+\frac{1}{2}, m_2, q; p-1>\\
&\;=\Big[-\frac{1}{4}\left((3J_1+m_1+\frac{5}{2})(3J_2+m_2+\frac{5}{2})+\frac{1}{3}(J_1-m_1-\frac{1}{2})(J_2-m_2-\frac{1}{2})\right)a^\dagger_{12}\alpha^\dagger_{23}a^\dagger_{34}\\
&\quad+\frac{1}{2}(J_1-m_1-\frac{1}{2})(3J_2+m_2+\frac{5}{2})\alpha^\dagger_{13}a^\dagger_{34}-\frac{1}{2}(3J_1+m_1+\frac{5}{2})(J_2-m_2-\frac{1}{2})a^\dagger_{12}\alpha^\dagger_{24}\\
&\quad+(J_1-m_1-\frac{1}{2})(J_2-m_2-\frac{1}{2})\alpha^\dagger_{14}\Big]\left(a^\dagger_{12}\right)^{J_1-m_1-\frac{3}{2}}\left(a^\dagger_{34}\right)^{J_2-m_2-\frac{3}{2}}|0>,\\
&\quad m_1=J_1-\frac{1}{2}, J_1-\frac{3}{2},\cdots,-(J_1+\frac{3}{2}),\quad m_2=J_2-\frac{1}{2}, J_2-\frac{3}{2},\cdots,-(J_2+\frac{3}{2}),\\
&|J_1-\frac{1}{2}, m_1, J_2+\frac{1}{2}, m_2, q; p-1>\\
&\;=\Big[\frac{1}{2}(3J_2+m_2+\frac{5}{2})\alpha^\dagger_{13}a^\dagger_{34}+\frac{1}{3}(2J_2+m_2+2)a^\dagger_{12}\alpha^\dagger_{23}a^\dagger_{34}\\
&\quad+(J_2-m_2-\frac{1}{2})(\alpha^\dagger_{14}-\frac{1}{2}a^\dagger_{12}\alpha^\dagger_{24})\Big]\left(a^\dagger_{12}\right)^{J_1-m_1-\frac{3}{2}}\left(a^\dagger_{34}\right)^{J_2-m_2-\frac{3}{2}}|0>,\quad J_1\geq\frac{1}{2},\\
&\quad m_1=J_1-\frac{3}{2}, J_1-\frac{5}{2},\cdots,-(J_1+\frac{1}{2}),\quad m_2=J_2-\frac{1}{2}, J_2-\frac{3}{2},\cdots,-(J_2+\frac{3}{2}).
\end{aligned}
\tag{III.3}
$$

The dimensions for these multiplets are $(2J_1)(2J_2)$, $(2J_1+2)(2J_2)$, $(2J_1+2)(2J_2+2)$ and $(2J_1)(2J_2+2)$, respectively.

The six level-2 states can be combined into 6 independent multiplets of $gl(2)\oplus gl(2)$ with highest weights $(J_1, J_2-1, q, p-2)$, $(J_1-1, J_2, q, p-2)$, $(J_1+1, J_2, q, p-2)$, $(J_1, J_2+1, q, p-2)$, $(J_1, J_2, q, p-2)$ and $(J_1, J_2, q, p-2)$, respectively:

$$
|J_1, m_1, J_2-1, m_2, q; p-2>=\alpha^\dagger_{24}\alpha^\dagger_{14}\left(a^\dagger_{12}\right)^{J_1-m_1-2}\left(a^\dagger_{34}\right)^{J_2-m_2-3}|0>
$$

$$+\frac{1}{2}\left[-\alpha_{23}^{\dagger}\alpha_{14}^{\dagger}+\frac{1}{6}\alpha_{23}^{\dagger}\alpha_{24}^{\dagger}a_{12}^{\dagger}+\alpha_{13}^{\dagger}\alpha_{24}^{\dagger}+\frac{1}{2}\alpha_{23}^{\dagger}\alpha_{13}^{\dagger}a_{34}^{\dagger}\right]\left(a_{12}^{\dagger}\right)^{J_1-m_1-2}\left(a_{34}^{\dagger}\right)^{J_2-m_2-2}|0>,$$

$$J_2\geq 1,\quad m_1=J_1-2,J_1-3,\cdots,-(J_1+2),\quad m_2=J_2-3,J_2-4,\cdots,-(J_2+1),$$

$$|J_1-1,m_1,J_2,m_2,q;p-2>=\alpha_{13}^{\dagger}\alpha_{14}^{\dagger}\left(a_{12}^{\dagger}\right)^{J_1-m_1-3}\left(a_{34}^{\dagger}\right)^{J_2-m_2-2}|0>$$

$$+\frac{1}{2}\left[\alpha_{23}^{\dagger}\alpha_{14}^{\dagger}+\frac{1}{6}\alpha_{23}^{\dagger}\alpha_{13}^{\dagger}a_{34}^{\dagger}+\alpha_{13}^{\dagger}\alpha_{24}^{\dagger}+\frac{1}{2}\alpha_{23}^{\dagger}\alpha_{24}^{\dagger}a_{12}^{\dagger}\right]\left(a_{12}^{\dagger}\right)^{J_1-m_1-2}\left(a_{34}^{\dagger}\right)^{J_2-m_2-2}|0>,$$

$$J_1\geq 1,\quad m_1=J_1-3,J_1-4,\cdots,-(J_1+1),\quad m_2=J_2-2,J_2-3,\cdots,-(J_2+2),$$

$$|J_1+1,m_1,J_2,m_2,q;p-2>$$

$$=\left[\frac{1}{2}[J_1-m_1-1+(3J_1+m_1+3)(3J_1+m_1+5)]\alpha_{23}^{\dagger}\alpha_{24}^{\dagger}\left(a_{12}^{\dagger}\right)^2\right.$$

$$+(J_1-m_1-1)(J_1-m_1-2)(\alpha_{13}^{\dagger}\alpha_{14}^{\dagger}+\frac{1}{12}a_{12}^{\dagger}\alpha_{23}^{\dagger}\alpha_{13}^{\dagger}a_{34}^{\dagger})$$

$$\left.-\frac{1}{2}(J_1-m_1-1)(3J_1+m_1+4)a_{12}^{\dagger}(\alpha_{13}^{\dagger}\alpha_{24}^{\dagger}+\alpha_{23}^{\dagger}\alpha_{14}^{\dagger})\right]\left(a_{12}^{\dagger}\right)^{J_1-m_1-3}\left(a_{34}^{\dagger}\right)^{J_2-m_2-2}|0>,$$

$$m_1=J_1-1,J_1-2,\cdots,-(J_1+3),\quad m_2=J_2-2,J_2-3,\cdots,-(J_2+2),$$

$$|J_1,m_1,J_2+1,m_2,q;p-2>$$

$$=\left[\frac{1}{4}[J_2-m_2-1+(3J_2+m_2+3)(3J_2+m_2+5)]\alpha_{23}^{\dagger}\alpha_{13}^{\dagger}\left(a_{34}^{\dagger}\right)^2\right.$$

$$+\frac{1}{2}(J_2-m_2-1)(3J_2+m_2+4)(\alpha_{23}^{\dagger}\alpha_{14}^{\dagger}-\alpha_{13}^{\dagger}\alpha_{24}^{\dagger})a_{34}^{\dagger}$$

$$\left.+(J_2-m_2-1)(J_2-m_2-2)(\alpha_{24}^{\dagger}\alpha_{14}^{\dagger}+\frac{1}{12}\alpha_{23}^{\dagger}\alpha_{24}^{\dagger}a_{12}^{\dagger}a_{34}^{\dagger})\right]$$

$$\times\left(a_{12}^{\dagger}\right)^{J_1-m_1-2}\left(a_{34}^{\dagger}\right)^{J_2-m_2-3}|0>,$$

$$m_1=J_1-2,J_1-3,\cdots,-(J_1+2),\quad m_2=J_2-1,J_2-2,\cdots,-(J_2+3),$$

$$|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{I}}=(J_2-m_2-2)\alpha_{24}^{\dagger}\alpha_{14}^{\dagger}\left(a_{12}^{\dagger}\right)^{J_1-m_1-2}\left(a_{34}^{\dagger}\right)^{J_2-m_2-3}|0>$$

$$+\left[\frac{1}{2}(J_2+m_2+2)(\alpha_{23}^{\dagger}\alpha_{14}^{\dagger}-\alpha_{13}^{\dagger}\alpha_{24}^{\dagger})+\frac{1}{12}(J_2-m_2-2)\alpha_{23}^{\dagger}\alpha_{24}^{\dagger}a_{12}^{\dagger}\right.$$

$$\left.-\frac{1}{4}(3J_2+m_2+2)\alpha_{23}^{\dagger}\alpha_{13}^{\dagger}a_{34}^{\dagger}\right]\left(a_{12}^{\dagger}\right)^{J_1-m_1-2}\left(a_{34}^{\dagger}\right)^{J_2-m_2-2}|0>,$$

$$m_1=J_1-2,J_1-3,\cdots,-(J_1+2),\quad m_2=J_2-1,J_2-2,\cdots,-(J_2+2),$$

$$|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{II}}=(J_1-m_1-2)\alpha_{13}^{\dagger}\alpha_{14}^{\dagger}\left(a_{12}^{\dagger}\right)^{J_1-m_1-3}\left(a_{34}^{\dagger}\right)^{J_2-m_2-2}|0>$$

$$+\left[-\frac{1}{2}(J_1+m_1+2)(\alpha_{13}^{\dagger}\alpha_{24}^{\dagger}+\alpha_{23}^{\dagger}\alpha_{14}^{\dagger})+\frac{1}{12}(J_1-m_1-2)\alpha_{23}^{\dagger}\alpha_{13}^{\dagger}a_{34}^{\dagger}\right.$$

$$\left.-\frac{1}{4}(3J_1+m_1+2)\alpha_{23}^{\dagger}\alpha_{24}^{\dagger}a_{12}^{\dagger}\right]\left(a_{12}^{\dagger}\right)^{J_1-m_1-2}\left(a_{34}^{\dagger}\right)^{J_2-m_2-2}|0>,$$

$$m_1=J_1-2,J_1-3,\cdots,-(J_1+2),\quad m_2=J_2-1,J_2-2,\cdots,-(J_2+2).\tag{III.4}$$

Notice that the last two multiplets, which have been denoted above by $|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{I}}$ and $|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{II}}$, respectively, have the same highest weight $(J_1,J_2,q,p-2)$. This means that multiplicity will in general appear in the $gl(2|2)\downarrow gl(2)\oplus gl(2)$ branching rule. It is easy to see from the above expressions that $|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{I}}\equiv 0$ when $J_2=0$ and $|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{II}}\equiv 0$ when $J_1=0$.

The dimensions for the first four multiplets are $(2J_1+1)(2J_2-1)$, $(2J_1-1)(2J_2+$

1), $(2J_1+3)(2J_2+1)$ and $(2J_1+1)(2J_2+3)$, respectively. The dimension for $|J_1,m_1,J_2,m_2,q;p-2>_{\bf I}$ is $(2J_1+1)(2J_2+1)$ if $J_2\neq 0$ and zero if $J_2=0$. Similarly, the dimension for $|J_1,m_1,J_2,m_2,q;p-2>_{\bf II}$ is $(2J_1+1)(2J_2+1)$ if $J_1\neq 0$ and zero if $J_1=0$.

Finally, the four level-3 states are combined into four independent multiplets of $gl(2)\oplus gl(2)$ with highest weights $(J_1-\frac{1}{2},J_2-\frac{1}{2},q,p-3)$, $(J_1+\frac{1}{2},J_2-\frac{1}{2},q,p-3)$, $(J_1-\frac{1}{2},J_2+\frac{1}{2},q,p-3)$ and $(J_1+\frac{1}{2},J_2+\frac{1}{2},q,p-3)$, respectively:

$$\begin{aligned}
&|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3>=\Big[(\alpha^\dagger_{13}+\frac{1}{2}a^\dagger_{12}\alpha^\dagger_{23})\alpha^\dagger_{24}\alpha^\dagger_{14}\\
&\quad+\frac{1}{2}\alpha^\dagger_{23}\alpha^\dagger_{13}(\alpha^\dagger_{14}a^\dagger_{34}+\frac{1}{3}a^\dagger_{12}\alpha^\dagger_{24}a^\dagger_{34})\Big]\left(a^\dagger_{12}\right)^{J_1-m_1-\frac{7}{2}}\left(a^\dagger_{34}\right)^{J_2-m_2-\frac{7}{2}}|0>,\\
&\quad J_1,J_2\geq\frac{1}{2},\quad m_1=J_1-\frac{7}{2},\cdots,-(J_1+\frac{5}{2}),\quad m_2=J_2-\frac{7}{2},\cdots,-(J_2+\frac{5}{2}),\\
&|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3>=\Big[-\frac{1}{2}(3J_1+m_1+\frac{9}{2})\alpha^\dagger_{23}\alpha^\dagger_{24}\alpha^\dagger_{14}a^\dagger_{12}\\
&\quad+(J_1-m_1-\frac{5}{2})(-\alpha^\dagger_{24}+\frac{1}{2}\alpha^\dagger_{23}a^\dagger_{34})\alpha^\dagger_{13}\alpha^\dagger_{14}\\
&\quad-\frac{1}{6}(5J_1+m_1+\frac{11}{2})\alpha^\dagger_{23}\alpha^\dagger_{13}\alpha^\dagger_{24}a^\dagger_{12}a^\dagger_{34}\Big]\left(a^\dagger_{12}\right)^{J_1-m_1-\frac{7}{2}}\left(a^\dagger_{34}\right)^{J_2-m_2-\frac{7}{2}}|0>,\\
&\quad J_2\geq\frac{1}{2},\quad m_1=J_1-\frac{5}{2},\cdots,-(J_1+\frac{7}{2}),\quad m_2=J_2-\frac{7}{2},\cdots,-(J_2+\frac{5}{2}),\\
&|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-3>=\Big[-\frac{1}{2}(3J_2+m_2+\frac{9}{2})\alpha^\dagger_{23}\alpha^\dagger_{13}\alpha^\dagger_{14}a^\dagger_{34}\\
&\quad+(J_2-m_2-\frac{5}{2})(\alpha^\dagger_{13}+\frac{1}{2}\alpha^\dagger_{23}a^\dagger_{12})\alpha^\dagger_{24}\alpha^\dagger_{14}\\
&\quad-\frac{1}{6}(5J_2+m_2+\frac{11}{2})\alpha^\dagger_{23}\alpha^\dagger_{13}\alpha^\dagger_{24}a^\dagger_{12}a^\dagger_{34}\Big]\left(a^\dagger_{12}\right)^{J_1-m_1-\frac{7}{2}}\left(a^\dagger_{34}\right)^{J_2-m_2-\frac{7}{2}}|0>,\\
&\quad J_1\geq\frac{1}{2},\quad m_1=J_1-\frac{7}{2},\cdots,-(J_1+\frac{5}{2}),\quad m_2=J_2-\frac{5}{2},\cdots,-(J_2+\frac{7}{2}),\\
&|J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-3>=\Big[\frac{1}{4}\Big((3J_1+m_1+\frac{9}{2})(3J_2+m_2+\frac{9}{2})\\
&\quad-\frac{1}{3}(J_1-m_1-\frac{5}{2})(J_2-m_2-\frac{5}{2})\Big)\,a^\dagger_{12}\alpha^\dagger_{23}\alpha^\dagger_{13}\alpha^\dagger_{24}a^\dagger_{34}\\
&\quad-\frac{1}{2}(J_1-m_1-\frac{5}{2})(3J_2+m_2+\frac{9}{2})\alpha^\dagger_{23}\alpha^\dagger_{13}\alpha^\dagger_{14}a^\dagger_{34}\\
&\quad-\frac{1}{2}(3J_1+m_1+\frac{9}{2})(J_2-m_2-\frac{5}{2})a^\dagger_{12}\alpha^\dagger_{23}\alpha^\dagger_{24}\alpha^\dagger_{14}\\
&\quad+(J_1-m_1-\frac{5}{2})(J_2-m_2-\frac{5}{2})\alpha^\dagger_{13}\alpha^\dagger_{24}\alpha^\dagger_{14}\Big]\left(a^\dagger_{12}\right)^{J_1-m_1-\frac{7}{2}}\left(a^\dagger_{34}\right)^{J_2-m_2-\frac{7}{2}}|0>,\\
&\quad m_1=J_1-\frac{5}{2},\cdots,-(J_1+\frac{7}{2}),\quad m_2=J_2-\frac{5}{2},\cdots,-(J_2+\frac{7}{2}).
\end{aligned}\tag{III.5}$$

The dimensions for these multiplets are $(2J_1)(2J_2)$, $(2J_1+2)(2J_2)$, $(2J_1)(2J_2+2)$ and $(2J_1+2)(2J_2+2)$, respectively.

The actions of the odd generators of $gl(2|2)$ on the $gl(2)\oplus gl(2)$ multiplets (III.1) and (III.3)-(III.5) can be computed by means of the free boson-fermion realization of the generators. In the following we list the actions of the odd simple generators. The actions

of odd non-simple generators can be easily obtained using the commutation relations.

First for the level-0 multiplet, we have the actions of the odd simple generators

$$\begin{aligned}
&\Gamma(E_{23})|J_1,m_1,J_2,m_2,q;p>=0,\\
&\Gamma(E_{32})|J_1,m_1,J_2,m_2,q;p>=\frac{1}{(2J_1+1)(2J_2+1)}\\
&\quad\times\Big[-(q+J_1-J_2)(J_1-m_1)(J_2-m_2)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-1>\\
&\quad-(q-J_1-J_2-1)(J_2-m_2)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-1>\\
&\quad-(q-J_1+J_2)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-1>\\
&\quad+(q+J_1+J_2+1)(J_1-m_1)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-1>\Big].
\end{aligned}\tag{III.6}$$

From (III.6) we see that when $q=J_1-J_2$ (resp. $-J_1+J_2$) the third (resp. first) term vanishes and if $q=J_1+J_2+1$ (resp. $-J_1-J_2-1$) then the second (resp. fourth) term disappears. This indicates that when $q=\pm(J_1-J_2),\pm(J_1+J_2+1)$ atypical representations arise (see next section for details).

For the four level-1 multiplets, we obtain the the following actions of the odd simple generators, after long algebraic manipulations,

$$\begin{aligned}
&\Gamma(E_{23})|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q,p-1>=-|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p>,\\
&\Gamma(E_{23})|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q,p-1>\\
&\quad=-(J_1+m_1+\frac{3}{2})|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p>,\\
&\Gamma(E_{23})|J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q,p-1>\\
&\quad=-(J_1+m_1+\frac{3}{2})(J_2+m_2+\frac{3}{2})|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p>,\\
&\Gamma(E_{23})|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q,p-1>\\
&\quad=-(J_2+m_2+\frac{3}{2})|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p>,\\
&\Gamma(E_{32})|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q,p-1>\\
&\quad=-\frac{J_2-m_2-\frac{3}{2}}{2J_2}(q-J_1-J_2-1)|J_1,m_1-\frac{1}{2},J_2-1,m_2-\frac{1}{2},q;p-2>\\
&\quad+\frac{q-J_1+J_2-1}{2J_2}|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>_{\mathbf{I}}\\
&\quad-\frac{q-J_1+J_2+1}{2J_2}|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>_{\mathbf{II}}\\
&\quad+\frac{J_1-m_1-\frac{3}{2}}{2J_1}(q+J_1+J_2+1)|J_1-1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>,
\end{aligned}$$

$$\begin{aligned}
&\Gamma(E_{32})|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q,p-1>\\
&\quad=\frac{(J_1-m_1-\frac{1}{2})(J_2-m_2-\frac{3}{2})}{2J_2}(q+J_1-J_2)|J_1,m_1-\frac{1}{2},J_2-1,m_2-\frac{1}{2},q;p-2>\\
&\quad-\frac{J_1-m_1-\frac{1}{2}}{2J_2}(q+J_1+J_2)|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>_{\mathbf{I}}\\
&\quad-\frac{J_1-m_1-\frac{1}{2}}{2(J_1+1)}(q+J_1+J_2+2)|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>_{\mathbf{II}}\\
&\quad+\frac{q-J_1+J_2}{2(J_1+1)}|J_1+1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>,\\
&\Gamma(E_{32})|J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q,p-1>\\
&\quad=-\frac{(J_1-m_1-\frac{1}{2})(J_2-m_2-\frac{1}{2})}{2(J_2+1)}(q+J_1-J_2-1)|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>_{\mathbf{I}}\\
&\quad+\frac{(J_1-m_1-\frac{1}{2})(J_2-m_2-\frac{1}{2})}{2(J_1+1)}(q+J_1-J_2+1)|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>_{\mathbf{II}}\\
&\quad-\frac{J_2-m_2-\frac{1}{2}}{2(J_1+1)}(q-J_1-J_2-1)|J_1+1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>\\
&\quad+\frac{J_1-m_1-\frac{1}{2}}{2(J_2+1)}(q+J_1+J_2+1)|J_1,m_1-\frac{1}{2},J_2+1,m_2-\frac{1}{2},q;p-2>,\\
&\Gamma(E_{32})|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q,p-1>\\
&\quad=-\frac{J_2-m_2-\frac{1}{2}}{2(J_2+1)}(q-J_1-J_2-2)|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>_{\mathbf{I}}\\
&\quad-\frac{J_2-m_2-\frac{1}{2}}{2J_1}(q-J_1-J_2)|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>_{\mathbf{II}}\\
&\quad+\frac{(J_1-m_1-\frac{3}{2})(J_2-m_2-\frac{1}{2})}{2J_1}(q+J_1-J_2)|J_1-1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-2>\\
&\quad+\frac{q-J_1+J_2}{2(J_2+1)}|J_1,m_1-\frac{1}{2},J_2+1,m_2-\frac{1}{2},q;p-2>\,. \qquad \text{(III.7)}
\end{aligned}$$

Similar to the level-1 case, we find after long algebraic computations that the actions of the odd simple generators on the six level-2 multiplets are given by

$$\begin{aligned}
&\Gamma(E_{23})|J_1,m_1,J_2-1,m_2,q;p-2>\\
&\quad=-\frac{J_1+m_1+2}{2J_1+1}|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad+\frac{1}{2J_1+1}|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-1>,\\
&\Gamma(E_{23})|J_1-1,m_1,J_2,m_2,q;p-2>\\
&\quad=\frac{J_2+m_2+2}{2J_2+1}|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad+\frac{1}{2J_1+1}|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-1>,
\end{aligned}$$

$$
\begin{aligned}
&\Gamma(E_{23})|J_1+1,m_1,J_2,m_2,q;p-2>\\
&\quad=\frac{(J_1+m_1+3)(J_2+m_2+2)}{2J_2+1}|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad-\frac{J_1+m_1+3}{2J_2+1}|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-1>,\\
&\Gamma(E_{23})|J_1,m_1,J_2+1,m_2,q;p-2>\\
&\quad=\frac{J_2+m_2+3}{2J_1+1}|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad+\frac{(J_1+m_1+2)(J_2+m_2+3)}{2J_1+1}|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-1>,\\
&\Gamma(E_{23})|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{I}}=\frac{1}{(2J_1+1)(2J_2+1)}\times\\
&\quad\Big[(J_2+1)(J_1+m_1+2)(J_2+m_2+2)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad-(J_2+1)(J_2+m_2+2)|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad-J_2|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad-J_2(J_1+m_1+2)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-1>\Big],\\
&\Gamma(E_{23})|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{II}}=\frac{1}{(2J_1+1)(2J_2+1)}\times\\
&\quad\Big[-(J_1+1)(J_1+m_1+2)(J_2+m_2+2)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad-J_1(J_2+m_2+2)|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad+J_1|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-1>\\
&\quad-(J_1+1)(J_1+m_1+2)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-1>\Big],\\
&\Gamma(E_{32})|J_1,m_1,J_2-1,m_2,q;p-2>\\
&\quad=\frac{J_1-m_1-2}{2J_1+1}(q+J_1+J_2+1)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad-\frac{q-J_1+J_2}{2J_1+1}|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>,\\
&\Gamma(E_{32})|J_1-1,m_1,J_2,m_2,q;p-2>\\
&\quad=\frac{J_2-m_2-2}{2J_2+1}(q-J_1-J_2-1)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad-\frac{q-J_1+J_2}{2J_2+1}|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>,\\
&\Gamma(E_{32})|J_1+1,m_1,J_2,m_2,q;p-2>=\frac{(J_1-m_1-1)(J_2-m_2-2)}{2J_2+1}\\
&\quad\times(q+J_1-J_2)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{J_1-m_1-1}{2J_2+1}(q+J_1+J_2+1)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>,\\
&\Gamma(E_{32})|J_1,m_1,J_2+1,m_2,q;p-2>=\frac{(J_1-m_1-2)(J_2-m_2-1)}{2J_1+1}\\
&\quad\times(q+J_1-J_2)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad-\frac{J_2-m_2-1}{2J_1+1}(q-J_1-J_2-1)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>,\\
&\Gamma(E_{32})|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{I}}=\frac{1}{(2J_1+1)(2J_2+1)}\left[(J_2+1)(J_1-m_1-2)\right.\\
&\quad\times(J_2-m_2-2)(q+J_1-J_2+1)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad-(J_2+1)(J_2-m_2-2)(q-J_1-J_2)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad+J_2(J_1-m_1-2)(q+J_1+J_2+2)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad\left.-J_2(q-J_1+J_2+1)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>\right],\\
&\Gamma(E_{32})|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{II}}=\frac{1}{(2J_1+1)(2J_2+1)}\left[(J_1+1)(J_1-m_1-2)\right.\\
&\quad\times(J_2-m_2-2)(q+J_1-J_2-1)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad+J_1(J_2-m_2-2)(q-J_1-J_2-2)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad-(J_1+1)(J_1-m_1-2)(q+J_1+J_2)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>\\
&\quad\left.-J_1(q-J_1+J_2-1)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>\right].
\end{aligned}
\tag{III.8}
$$

The actions of the odd simple generators on the four level-3 multiplets can be obtained in a similar way. We list the results as follows:

$$
\begin{aligned}
&\Gamma(E_{23})|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q,p-3>\\
&\quad=\frac{J_2+m_2+\frac{5}{2}}{2J_2}|J_1,m_1+\frac{1}{2},J_2-1,m_2+\frac{1}{2},q;p-2>\\
&\quad+\frac{1}{2J_2}|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>_{\mathbf{I}}+\frac{1}{2J_1}|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>_{\mathbf{II}}\\
&\quad+\frac{J_1+m_1+\frac{5}{2}}{2J_1}|J_1-1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>,\\
&\Gamma(E_{23})|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q,p-3>\\
&\quad=-\frac{(J_1+m_1+\frac{7}{2})(J_2+m_2+\frac{5}{2})}{2J_2}|J_1,m_1+\frac{1}{2},J_2-1,m_2+\frac{1}{2},q;p-2>\\
&\quad+(J_1+m_1+\frac{7}{2})\left[-\frac{1}{2J_2}|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>_{\mathbf{I}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2(J_1+1)}|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>_{\mathbf{II}}\Big]\\
&+\frac{1}{2(J_1+1)}|J_1+1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>,\\
&\Gamma(E_{23})|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q,p-3>\\
&=(J_2+m_2+\frac{7}{2})\left[\frac{1}{2(J_2+1)}|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>_{\mathbf{I}}\right.\\
&\left.-\frac{1}{2J_1}|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>_{\mathbf{II}}\right]\\
&-\frac{(J_1+m_1+\frac{5}{2})(J_2+m_2+\frac{7}{2})}{2J_1}|J_1-1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>\\
&+\frac{1}{2(J_2+1)}|J_1,m_1+\frac{1}{2},J_2+1,m_2+\frac{1}{2},q;p-2>,\\
&\Gamma(E_{23})|J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q,p-3>=-(J_1+m_1+\frac{7}{2})(J_2+m_2+\frac{7}{2})\\
&\times\left[\frac{1}{2(J_2+1)}|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>_{\mathbf{I}}\right.\\
&\left.+\frac{1}{2(J_1+1)}|J_1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>_{\mathbf{II}}\right]\\
&-\frac{J_2+m_2+\frac{7}{2}}{2(J_1+1)}|J_1+1,m_1+\frac{1}{2},J_2,m_2+\frac{1}{2},q;p-2>\\
&-\frac{J_1+m_1+\frac{7}{2}}{2(J_2+1)}|J_1,m_1+\frac{1}{2},J_2+1,m_2+\frac{1}{2},q;p-2>,\\
&\Gamma(E_{32})|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q,p-3>\\
&=(q-J_1+J_2)|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-4>,\\
&\Gamma(E_{32})|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q,p-3>\\
&=(q+J_1+J_2+1)(J_1-m_1-\frac{5}{2})|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-4>,\\
&\Gamma(E_{32})|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q,p-3>\\
&=(q-J_1-J_2-1)(J_2-m_2-\frac{5}{2})|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-4>,\\
&\Gamma(E_{32})|J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q,p-3>\\
&=(q+J_1-J_2)(J_1-m_1-\frac{5}{2})(J_2-m_2-\frac{5}{2})|J_1,m_1-\frac{1}{2},J_2,m_2-\frac{1}{2},q;p-4>.
\end{aligned}
\tag{III.9}
$$

Finally, the actions of the odd simple generators on the level-4 multiplet are

$$
\Gamma(E_{23})|J_1,m_1,J_2,m_2,q;p-4>=\frac{1}{(2J_1+1)(2J_2+1)}
$$

$$\times\Big[(J_1+m_1+4)(J_2+m_2+4)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-3>$$
$$+(J_2+m_2+4)|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q,p-3>$$
$$+(J_1+m_1+4)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-3>$$
$$+|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q,p-3>\Big],$$
$$\Gamma(E_{32})|J_1,m_1,J_2,m_2,q;p-4>=0. \tag{III.10}$$

Summarizing, we have obtained 16 independent multiplets, (III.1) and (III.3)–(III.4), of $gl(2)\oplus gl(2)$ which span finite-dimensional representations of $gl(2|2)$. For generic q, these multiplets span irreducible typical representations of $gl(2|2)$ of dimension $16(2J_1+1)(2J_2+1)$. Denote by $\pi_{(J_1,J_2,q,p)}$ and $\sigma_{(J_1,J_2,q,p)}$ the $gl(2|2)$ and $gl(2)\oplus gl(2)$ representations with highest weight (J_1,J_2,q,p), respectively. Then the $gl(2|2)\downarrow gl(2)\oplus gl(2)$ branching rule for generic q is given by

$$\begin{aligned}\pi_{(J_1,J_2,q,p)} &= \sigma_{(J_1,J_2,q,p)}\oplus\sigma_{(J_1-1/2,J_2-1/2,q,p-1)}\oplus\sigma_{(J_1+1/2,J_2-1/2,q,p-1)}\\ &\oplus\sigma_{(J_1+1/2,J_2+1/2,q,p-1)}\oplus\sigma_{(J_1-1/2,J_2+1/2,q,p-1)}\oplus\sigma_{(J_1,J_2-1,q,p-2)}\\ &\oplus\sigma_{(J_1-1,J_2,q,p-2)}\oplus\sigma_{(J_1+1,J_2,q,p-2)}\oplus\sigma_{(J_1,J_2+1,q,p-2)}\oplus 2\times\sigma_{(J_1,J_2,q,p-2)}\\ &\oplus\sigma_{(J_1-1/2,J_2-1/2,q,p-3)}\oplus\sigma_{(J_1+1/2,J_2-1/2,q,p-3)}\oplus\sigma_{(J_1-1/2,J_2+1/2,q,p-3)}\\ &\oplus\sigma_{(J_1+1/2,J_2+1/2,q,p-3)}\oplus\sigma_{(J_1,J_2,q,p-4)}.\end{aligned} \tag{III.11}$$

Some remarks are in order. Firstly, ireducible representations are obtained as submodules (not subquotients) of the super-Fock space generated by $\{a_{ij},a^\dagger_{ij},\alpha_{ij},\alpha^\dagger_{ij}\}$. This is because the $gl(2|2)$-module structure of the super-Fock space is the contragredient dual of the Verma model over $gl(2|2)$. Secondly, as $|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{I}}\equiv 0$ when $J_2=0$ and $|J_1,m_1,J_2,m_2,q;p-2>_{\mathbf{II}}\equiv 0$ when $J_1=0$, thus if $J_1=0$ or $J_2=0$ only one copy of $\sigma_{(J_1,J_2,q,p-2)}$ remains in the above branching rule. In particular, when $J_1=0=J_2$ which corresponds to the 16-dimensional typical representation of $gl(2|2)$, $\sigma_{(J_1,J_2,q,p-2)}$ disappears and the branching rule becomes

$$\begin{aligned}\pi_{(0,0,q,p)} &= \sigma_{(0,0,q,p)}\oplus\sigma_{(1/2,1/2,q,p-1)}\oplus\sigma_{(1,0,q,p-2)}\\ &\oplus\sigma_{(0,1,q,p-2)}\oplus\sigma_{(1/2,1/2,q,p-3)}\oplus\sigma_{(0,0,q,p-4)}\end{aligned} \tag{III.12}$$

or $\underline{16}=\underline{1}\oplus\underline{4}\oplus\underline{3}\oplus\underline{3}\oplus\underline{4}\oplus\underline{1}$.

IV Atypical Representations of $gl(2|2)$

We have different types of atypical representations of $gl(2|2)$. From the actions of the odd generators on the $gl(2)\oplus gl(2)$ multiplets, we see that when $q=\pm(J_1-J_2),\pm(J_1+J_2+1)$, the representations become atypical. The Casimir for such representations vanishes, and yet they are not the trivial one-dimensional representation.

IV.1 Atypical representation corresponding to $q = J_1 - J_2$

Case 1. $q = J_1 - J_2,\ \ J_1 \neq J_2$:

Let us introduce the following independent combinations:

$$\begin{aligned}
|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{sym1}} &= J_1|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{I}} \\
&\quad + J_2|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{II}}, \\
|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{asym1}} &= J_1|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{I}} \\
&\quad - J_2|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{II}} \qquad \text{(IV.1)}
\end{aligned}$$

for $J_1 \neq 0, J_2 \neq 0$. When $J_1 = 0$ or $J_2 = 0$, we let $|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{sym1}} \equiv 0$ and

$$|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{asym1}} = \begin{cases} |J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{I}} & \text{if } J_1 = 0, \\ |J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{II}} & \text{if } J_2 = 0. \end{cases} \qquad \text{(IV.2)}$$

It can be shown from the actions of odd generators that when $q = J_1 - J_2$,

$$\begin{aligned}
&\Gamma(E_{23})|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{sym1}} = \frac{1}{(2J_1+1)(2J_2+1)} \times \\
&\quad \Big[(J_1 - J_2)(J_1 + m_1 + 2)(J_2 + m_2 + 2)|J_1 - \frac{1}{2}, m_1 + \frac{1}{2}, J_2 - \frac{1}{2}, m_2 + \frac{1}{2}, q; p-1> \\
&\quad - J_1(2J_2+1)(J_2 + m_2 + 2)|J_1 + \frac{1}{2}, m_1 + \frac{1}{2}, J_2 - \frac{1}{2}, m_2 + \frac{1}{2}, q; p-1> \\
&\quad - (2J_1+1)J_2(J_1 + m_1 + 2)|J_1 - \frac{1}{2}, m_1 + \frac{1}{2}, J_2 + \frac{1}{2}, m_2 + \frac{1}{2}, q; p-1>\Big] \qquad \text{(IV.3)}
\end{aligned}$$

which does not contain the multiplet $|J_1 + \frac{1}{2}, m_1, J_2 + \frac{1}{2}, m_2, q; p-1>$ and

$$\begin{aligned}
\Gamma(E_{32})|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{sym1}} &= \frac{(J_1 - J_2)(4J_1J_2 + 2J_1 + 2J_2 + 1)}{(2J_1+1)(2J_2+1)} \\
&\quad \times (J_1 - m_1 - 2)(J_2 - m_2 - 2)|J_1 - \frac{1}{2}, m_1 - \frac{1}{2}, J_2 - \frac{1}{2}, m_2 - \frac{1}{2}, q; p-3> \qquad \text{(IV.4)}
\end{aligned}$$

Thus when $q = J_1 - J_2$, if one starts with the level-0 state $|J_1, m_1, J_2, m_2, q; p>$ then we find using the actions (III.6-III.10) that the following $gl(2) \oplus gl(2)$ multiplets

$$\begin{aligned}
&|J_1 + \frac{1}{2}, m_1, J_2 + \frac{1}{2}, m_2, q; p-1>, \\
&|J_1 + 1, m_1, J_2, m_2, q, p-2>, \qquad |J_1, m_1, J_2 + 1, m_2, q, p-2>, \\
&|J_1, m_1, J_2, m_2, q, p-2>_{\mathbf{asym1}}, \qquad |J_1 + \frac{1}{2}, m_1, J_2 - \frac{1}{2}, m_2, q; p-3>, \\
&|J_1 - \frac{1}{2}, m_1, J_2 + \frac{1}{2}, m_2, q; p-3>, \qquad |J_1 + \frac{1}{2}, m_1, J_2 + \frac{1}{2}, m_2, q; p-3>, \\
&|J_1, m_1, J_2, m_2, q, p-4> \qquad \text{(IV.5)}
\end{aligned}$$

disappear, and only the following multiplets

$$|J_1, m_1, J_2, m_2, q, p>,$$

$$
\begin{aligned}
&|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>, \qquad |J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>,\\
&|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>,\\
&|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym1}}, \qquad |J_1-1,m_1,J_2,m_2,q,p-2>,\\
&|J_1,m_1,J_2-1,m_2,q,p-2>, \qquad |J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3> \qquad \text{(IV.6)}
\end{aligned}
$$

remain. They form irreducible atypical representations of $gl(2|2)$ of dimension $8[(2J_1+1)J_2+J_1(2J_2+1)]$. So the $gl(2|2)\downarrow gl(2)\oplus gl(2)$ branching rule for $q=J_1-J_2$ is given by

$$
\begin{aligned}
\pi_{(J_1,J_2,q,p)} &= \sigma_{(J_1,J_2,q,p)}\oplus\sigma_{(J_1-\frac{1}{2},J_2-\frac{1}{2},q,p-1)}\oplus\sigma_{(J_1+\frac{1}{2},J_2+\frac{1}{2},q,p-1)}\\
&\oplus\sigma_{(J_1-\frac{1}{2},J_2+\frac{1}{2},q,p-1)}\oplus\sigma_{(J_1,J_2,q,p-2)}\oplus\sigma_{(J_1-1,J_2,q,p-2)}\\
&\oplus\sigma_{(J_1,J_2-1,q,p-2)}\oplus\sigma_{(J_1-\frac{1}{2},J_2-\frac{1}{2},q,p-3)}. \qquad \text{(IV.7)}
\end{aligned}
$$

It should be understood here that $\sigma_{(J_1,J_2,q,p-2)}$ disappears when $J_1=0$ or $J_2=0$.

Case 2. $q=J_1-J_2,\ J_1=J_2$ so that $q=0$:

In this case, we define the independent combinations:

$$
\begin{aligned}
|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym1'}} &= |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}}\\
&+|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}},\\
|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym1'}} &= |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}}\\
&-|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}}\ . \qquad \text{(IV.8)}
\end{aligned}
$$

Both $|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym1'}}$ and $|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym1'}}$ vanish if $J_1=0=J_2$. Then it is easily shown that $\Gamma(E_{23})|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym1'}}$ does not contain $|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>$ and $|J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>$, and $\Gamma(E_{32})|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym1'}}=0$. Thus only the following multiplets

$$
\begin{aligned}
&|J_1,m_1,J_2,m_2,q,p>,\\
&|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>, \qquad |J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>,\\
&|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym1'}} \qquad \text{(IV.9)}
\end{aligned}
$$

survive, and they give irreducible atypical representations of dimension $4[(2J_1+1)(2J_2+1)-1/2]$ if $J_1=J_2\neq 0$ and the trivial one-deimsional representaion if $J_1=0=J_2$ (for which the last three multiplets in (IV.9) disappear).

Case 3. Lowest weight (indecomposable) Kac modules:

Other types of atypical representations when $q=J_1-J_2$ are not irreducible. One such type of representations are obtained by starting with the level-4 state $|J_1,m_1,J_2,m_2,q;p-$

$4>$. These representations contain all 16 multiplets and a non-separable invariant subspace provided by the multiplets (IV.6) [or (IV.9) when $J_1 = J_2$]. These representations are not fully reducible (i.e. indecomposable) and have dimension $16(2J_1+1)(2J_2+1)$.

IV.2 Atypical representations corresponding to $q = -J_1 + J_2$

The case where $J_1 = J_2$ so that $q = 0$ is the same as Case 2 of the last subsection. So in this subsection we only consider the $J_1 \neq J_2$ case.

1. Irreducible representations:

Let us introduce the following independent combinations:

$$\begin{aligned}
|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym2}} &= (J_1+1)|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}}\\
&\quad +(J_2+1)|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}},\\
|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym2}} &= (J_1+1)|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}}\\
&\quad -(J_2+1)|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}} \qquad \text{(IV.10)}
\end{aligned}$$

for $J_1 \neq 0, J_2 \neq 0$, and let

$$|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym2}} = \begin{cases} |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}} & \text{if } J_1 = 0,\\ |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}} & \text{if } J_2 = 0.\end{cases} \qquad \text{(IV.11)}$$

and $|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym2}} = 0$ if $J_1 = 0$ or $J_2 = 0$.

Similar to the $q = J_1 - J_2$ case, we may show that when $q = -J_1 + J_2$,

$$\begin{aligned}
&\Gamma(E_{23})|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym2}} = \frac{1}{(2J_1+1)(2J_2+1)} \times\\
&\quad\Big[-(2J_1+1)(J_2+1)(J_2+m_2+2)|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q;p-1>\\
&\quad+(J_1-J_2)|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q;p-1>\\
&\quad-(J_1+1)(2J_2+1)(J_1+m_1+2)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q;p-1>\Big]
\end{aligned} \qquad \text{(IV.12)}$$

which is independent of $|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>$ and

$$\begin{aligned}
&\Gamma(E_{32})|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym2}} = \frac{(J_1-J_2)(4J_1J_2+2J_1+2J_2+1)}{(2J_1+1)(2J_2+1)}\\
&\quad\times(J_1-m_1-2)(J_2-m_2-2)|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q;p-3>
\end{aligned} \qquad \text{(IV.13)}$$

Thus when $q = -J_1 + J_2$, if one starts with the level-0 state then by the actions (III.6-III.10) one finds that the following $gl(2) \oplus gl(2)$ multiplets

$$|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>,$$

$$|J_1,m_1,J_2-1,m_2,q,p-2>,\quad |J_1-1,m_1,J_2,m_2,q,p-2>,$$
$$|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym2}},\quad |J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3>,$$
$$|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3>,\quad |J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-3>,$$
$$|J_1,m_1,J_2,m_2,q,p-4> \tag{IV.14}$$

drop out of the basis, and only the following multiplets

$$|J_1,m_1,J_2,m_2,q,p>,$$
$$|J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>,\quad |J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>,$$
$$|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>,$$
$$|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym2}},\quad |J_1+1,m_1,J_2,m_2,q,p-2>,$$
$$|J_1,m_1,J_2+1,m_2,q,p-2>,\quad |J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-3> \tag{IV.15}$$

survive. They form irreducible atypical representations of $gl(2|2)$ of dimension $8[(J_1+1)(2J_2+1)+(2J_1+1)(J_2+1)]$. The branching rule in this case (i.e. $q=-J_1+J_2$) becomes

$$\begin{aligned}\pi_{(J_1,J_2,q,p)} &= \sigma_{(J_1,J_2,q,p)}\oplus\sigma_{(J_1+\frac{1}{2},J_2+\frac{1}{2},q,p-1)}\oplus\sigma_{(J_1+\frac{1}{2},J_2-\frac{1}{2},q,p-1)}\\ &\oplus\sigma_{(J_1-\frac{1}{2},J_2+\frac{1}{2},q,p-1)}\oplus\sigma_{(J_1,J_2,q,p-2)}\oplus\sigma_{(J_1+1,J_2,q,p-2)}\\ &\oplus\sigma_{(J_1,J_2+1,q,p-2)}\oplus\sigma_{(J_1+\frac{1}{2},J_2+\frac{1}{2},q,p-3)}.\end{aligned} \tag{IV.16}$$

2. Lowest weight (indecomposable) Kac modules:

If one starts with the level-4 state, then one gets atypical representations which are not irreducible. In such representations, all 16 multiplets appear but there exists a non-separable invariant superspace generated by multiplets (IV.15). These representations are indecomposable and have dimension $16(2J_1+1)(2J_2+1)$.

IV.3 Atypical representations corresponding to $q=J_1+J_2+1$

1. Irreducible representations:

Let us introduce the following independent combinations for $J_1\neq 0, J_2\neq 0$,

$$\begin{aligned}|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym3}} &= J_1\,|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}}\\ &+(J_2+1)|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}},\\ |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym3}} &= J_1\,|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}}\\ &-(J_2+1)|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}}\,.\end{aligned} \tag{IV.17}$$

We let

$$
\begin{aligned}
|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym3}} &= \begin{cases} |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}} & \text{if } J_1=0, \\ 0 & \text{if } J_2=0, \end{cases} \\
|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym3}} &= \begin{cases} 0 & \text{if } J_1=0, \\ |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}} & \text{if } J_2=0. \end{cases}
\end{aligned}
\tag{IV.18}
$$

It can be seen from the actions of odd generators that when $q=J_1+J_2+1$,

$$
\begin{aligned}
\Gamma(E_{23})|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym3}} &= \frac{1}{(2J_1+1)(2J_2+1)}\Big[(2J_1+1)(J_2+1) \\
&\times (J_2+m_2+2)(J_2+m_2+2)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q;p-1> \\
&- J_1(2J_2+1)|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q;p-1> \\
&+(J_1+J_2+1)(J_1+m_1+2)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q;p-1>\Big]
\end{aligned}
\tag{IV.19}
$$

which does not contain the multiplet $|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>$ and

$$
\begin{aligned}
\Gamma(E_{32})|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym3}} &= \frac{(J_1+J_2+1)(4J_1J_2+J_1+J_2)+(J_1+J_2)^2}{(2J_1+1)(2J_2+1)} \\
&\times (J_1-m_1-2)|J_1-\frac{1}{2},m_1-\frac{1}{2},J_2+\frac{1}{2},m_2-\frac{1}{2},q;p-3>\,.
\end{aligned}
\tag{IV.20}
$$

Then similar to previous cases, when $q=J_1+J_2+1$, the following $gl(2)\oplus gl(2)$ multiplets

$$
\begin{aligned}
&|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>, \\
&|J_1,m_1,J_2-1,m_2,q,p-2>, \quad |J_1+1,m_1,J_2,m_2,q,p-2>, \\
&|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym3}}, \quad |J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3>, \\
&|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3>, \quad |J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-3>, \\
&|J_1,m_1,J_2,m_2,q,p-4>
\end{aligned}
\tag{IV.21}
$$

disappear, and only the following multiplets

$$
\begin{aligned}
&|J_1,m_1,J_2,m_2,q,p>, \\
&|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>, \quad |J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>, \\
&|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>, \\
&|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym3}}, \quad |J_1-1,m_1,J_2,m_2,q,p-2>, \\
&|J_1,m_1,J_2+1,m_2,q,p-2>, \quad |J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-3>
\end{aligned}
\tag{IV.22}
$$

remain. They constitute irreducible atypical representations of $gl(2|2)$ of dimension $8[(2J_1+1)(J_2+1)+J_1(2J_2+1)]$. The branching rule in this case (i.e. $q=J_1+J_2+1$)

reads

$$\begin{aligned}\pi_{(J_1,J_2,q,p)} &= \sigma_{(J_1,J_2,q,p)} \oplus \sigma_{(J_1-\frac{1}{2},J_2-\frac{1}{2},q,p-1)} \oplus \sigma_{(J_1+\frac{1}{2},J_2+\frac{1}{2},q,p-1)}\\ &\oplus\sigma_{(J_1-\frac{1}{2},J_2+\frac{1}{2},q,p-1)} \oplus \sigma_{(J_1,J_2,q,p-2)} \oplus \sigma_{(J_1-1,J_2,q,p-2)}\\ &\oplus\sigma_{(J_1,J_2+1,q,p-2)} \oplus \sigma_{(J_1-\frac{1}{2},J_2+\frac{1}{2},q,p-3)}. \end{aligned} \tag{IV.23}$$

Here one should keep in mind that $\sigma_{(J_1,J_2,q,p-2)}$ disappears if $J_1 = 0$.

2. Lowest weight (indecomposable) Kac representations:

Similar to the previous cases, if one retains all 16 multiplets, then one gets lowest weight (indecomposable) Kac representations of $16(2J_1+1)(2J_2+1)$ which contain an invariant but non-separable subspace provided by multiplets (IV.22).

IV.4 Atypical representations corresponding to $q = -J_1 - J_2 - 1$

1. Irreducible representations:

In this case, we introduce the following independent combinations for $J_1 \neq 0, J_2 \neq 0$,

$$\begin{aligned}|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym4}} &= (J_1+1)|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}}\\ &\quad +J_2\,|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}},\\ |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym4}} &= (J_1+1)|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}}\\ &\quad -J_2\,|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}} \end{aligned} \tag{IV.24}$$

and let

$$\begin{aligned}|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{sym4}} &= \begin{cases} 0 & \text{if } J_1=0,\\ |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{II}} & \text{if } J_2=0,\end{cases}\\ |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym4}} &= \begin{cases} |J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{I}} & \text{if } J_1=0,\\ 0 & \text{if } J_2=0.\end{cases}\end{aligned} \tag{IV.25}$$

It can be seen from the actions of odd generators that when $q = -J_1 - J_2 - 1$,

$$\begin{aligned}\Gamma(E_{23})|J_1,m_1,J_2,m_2,q,p-2>_{\mathbf{asym4}} &= \frac{1}{(2J_1+1)(2J_2+1)}\Big[(J_1+1)(2J_2+1)\\ &\times(J_2+m_2+2)(J_2+m_2+2)|J_1-\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q;p-1>\\ &-(J_1+J_2+1)(J_2+m_2+2)|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2-\frac{1}{2},m_2+\frac{1}{2},q;p-1>\\ &-(2J_1+1)J_2|J_1+\frac{1}{2},m_1+\frac{1}{2},J_2+\frac{1}{2},m_2+\frac{1}{2},q;p-1>\Big]\end{aligned} \tag{IV.26}$$

which has no dependence on the multiplet $|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>$ and

$$\Gamma(E_{32})|J_1,m_1,J_2,m_2,q,p-2>_{\bf asym4}=-\frac{(J_1+J_2+1)(4J_1J_2+J_1+J_2)+(J_1+J_2)^2}{(2J_1+1)(2J_2+1)}$$
$$\times(J_2-m_2-2)|J_1+\frac{1}{2},m_1-\frac{1}{2},J_2-\frac{1}{2},m_2-\frac{1}{2},q;p-3>. \qquad \text{(IV.27)}$$

Thus when $q=-J_1-J_2-1$, the following $gl(2)\oplus gl(2)$ multiplets

$$\begin{array}{l}
|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>,\\
|J_1-1,m_1,J_2,m_2,q,p-2>,\quad |J_1,m_1,J_2+1,m_2,q,p-2>,\\
|J_1,m_1,J_2,m_2,q,p-2>_{\bf sym4},\quad |J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3>,\\
|J_1-\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-3>,\quad |J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-3>,\\
|J_1,m_1,J_2,m_2,q,p-4>
\end{array} \qquad \text{(IV.28)}$$

drop out, and only the following multiplets

$$\begin{array}{l}
|J_1,m_1,J_2,m_2,q,p>,\\
|J_1-\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>,\quad |J_1+\frac{1}{2},m_1,J_2+\frac{1}{2},m_2,q;p-1>,\\
|J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-1>,\\
|J_1,m_1,J_2,m_2,q,p-2>_{\bf asym4},\quad |J_1,m_1,J_2-1,m_2,q,p-2>,\\
|J_1+1,m_1,J_2,m_2,q,p-2>,\quad |J_1+\frac{1}{2},m_1,J_2-\frac{1}{2},m_2,q;p-3>
\end{array} \qquad \text{(IV.29)}$$

remain. They give irreducible atypical representations of $gl(2|2)$ of dimension $8[(J_1+1)(2J_2+1)+(2J_1+1)J_2]$. In this case the branching rule becomes

$$\begin{aligned}
\pi_{(J_1,J_2,q,p)} &= \sigma_{(J_1,J_2,q,p)}\oplus\sigma_{(J_1-\frac{1}{2},J_2-\frac{1}{2},q,p-1)}\oplus\sigma_{(J_1+\frac{1}{2},J_2+\frac{1}{2},q,p-1)}\\
&\oplus\sigma_{(J_1+\frac{1}{2},J_2-\frac{1}{2},q,p-1)}\oplus\sigma_{(J_1,J_2,q,p-2)}\oplus\sigma_{(J_1,J_2-1,q,p-2)}\\
&\oplus\sigma_{(J_1+1,J_2,q,p-2)}\oplus\sigma_{(J_1+\frac{1}{2},J_2-\frac{1}{2},q,p-3)}. \qquad \text{(IV.30)}
\end{aligned}$$

Here it should be understood that $\sigma_{(J_1,J_2,q,p-2)}$ is not in the branching rule if $J_2=0$.

2. Lowest weight (indecomposable) Kac representations:

As before, other types of atypical representations are not irreducible. These representations contain all 16 multiplets which contain a non-separable invariant subspace generated by multiplets (IV.29). They are lowest weight (indecomposable) Kac representations of dimension $16(2J_1+1)(2J_2+1)$.

V Conclusions and Discussions

In this article we have applied the super coherent state method to the construction of the free boson-fermion realization and representations of the non-semisimple superalgebra

$gl(2|2)$ in the standard basis. The representations are constructed out of the $gl(2) \oplus gl(2)$ particle states in the super-Fock space.

As mentioned in the introduction, superalgebras and their corresponding non-unitary CFTs emerge in the supersymmetric treatment to disordered systems and the integer quantum Hall plateaus. In such a treatment, primary fields play an important role in the computation of critical properties of the disordered systems. The results obtained in this paper now make possible the construction of all primary fields of the $gl(2|2)$ non-unitary CFT in terms of free fields [23]. This is under investigation and results will be presented elsewhere.

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